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Approximation of the Eigenvalues for Heat Transfer in Laminar Tube Slip Flow

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FOR convective heat transfer in laminar continuum tube flow with uniform wall heat flux, Sellars et al [1] have obtained asymptotic formulas for the eigenvalues and coefficients through a generalization of constant wall temperature results An improved and more direct treatment has been presented by Dzung [2]

The advent of space flight has brought about increased interest in the heat transfer to low-density gas flow in tubes Sparrow and Lin[3] have considered the fully developed heat transfer in circular tubes under slip-flow conditions It is of interest to determine the possible application of the method of Sellars et al to laminar tube slip flow

We wish to find solutions of

$$(d/d\eta)[\eta(dR_n/d\eta)] + \lambda_n(2f)\eta R_n = 0 \tag{1}$$

subject to

$$R_n(0) = 1 \tag{2}$$

and

$$[(dR_n/d\eta)] = 0 \text{ at } \eta = 0 \text{ and } \eta = 1 \tag{3}$$

where η is the dimensionless radial distance (r/r_0), $f(\eta)$ is the dimensionless velocity distribution,

$$f(\eta) \equiv u(\eta)/\bar{u} = 2[1 - \eta^2 + 4\alpha]/[1 + 8\alpha] \tag{4}$$

and $\alpha \equiv (\xi_u/d)$ The function R_n represents the radial temperature distribution in the thermal entrance region, λ_n is the eigenvalue, and ξ_u is the velocity slip coefficient [3] The velocity distribution, as given in Eq (4), assumes that thermal creep is negligible

In accordance with the method of Sellars et al, a solution of the form

$$R(\eta) = \exp[g(\eta)] \tag{5}$$

is considered, where

$$g = \lambda^{1/2}g_0 + g_1 + (g_2/\lambda^{1/2}) + \tag{6}$$

and, since λ is assumed to be large, only the first two terms of the foregoing series are retained It can be shown that R is given from Eqs (5) and (6) as

$$R = \left\{A \exp\left[i(\lambda)^{1/2}\int_0^\eta (2f)^{1/2}d\eta\right] + B \exp\left[-i(\lambda)^{1/2}\int_0^\eta (2f)^{1/2}d\eta\right]\right\} \Big/ \eta^{1/2}(2f)^{1/4} \tag{7}$$

excluding the singular point $\eta = 0$ It should be noted that, for continuum flow ($\alpha = 0$), a singularity also exists at $\eta = 1$, since $[f(1)]_{\alpha=0} = 0$ This has required the development of an alternate solution valid near $\eta = 1$ [1 2] For slip flow, no singularity exists at $\eta = 1$, since $[f(1)]_{\alpha\neq 0} = 8\alpha/(1 + 8\alpha) = u/\bar{u}$, where u is the slip velocity

The coefficients A and B are determined from continuation of Eq (7) to the central zone $\eta \approx 0$, where $R(\eta)$ can be approximated by a Bessel function

$$R(\eta) \approx J_0\{[2\lambda f(0)]^{1/2}\eta\} \qquad \eta^2 \ll 1 \tag{8}$$

where $f(0) = 2(1 + 4\alpha)/(1 + 8\alpha) = u/\bar{u}$ From the asymptotic expression of Bessel functions the coefficients are determined so that

$$R(\eta) = (\pi\eta)^{-1/2}(2/\lambda f)^{1/4}\cos[\lambda^{1/2}I - \pi/4] \tag{9}$$

where

$$I \equiv \int_0^\eta (2f)^{1/2}d\eta = \frac{\eta[1 + 4\alpha - \eta^2]^{1/2} + (1 + 4\alpha)\arcsin[\eta/(1 + 4\alpha)^{1/2}]}{(1 + 8\alpha)^{1/2}} \tag{10}$$

The slope of $R(\eta)$ at the wall is found by differentiating Eq (9) and setting $\eta = 1$; the result is

$$R'(1) = (8\pi)^{-1/2}(1 + 8\alpha)^{1/4}(4\alpha)^{-5/4}\lambda^{-1/4}[(E + F\gamma)\cos\gamma + (E - F\gamma)\sin\gamma] \tag{11}$$

where

$$\gamma \equiv \lambda^{1/2}I_1 \qquad I_1 \equiv \int_0^1 (2f)^{1/2}d\eta \qquad E \equiv 1 - 4\alpha$$

$$F \equiv 4(4\alpha)^{3/2}/\{(4\alpha)^{1/2} + (1 + 4\alpha)\arcsin[1/(1 + 4\alpha)^{1/2}]\}$$

Setting $R'(1) = 0$ yields a series of eigenvalues λ_n with the corresponding eigenfunction R_n as roots of the equation

$$\tan\gamma_n = (F\gamma_n + E)/(F\gamma_n - E) \tag{12}$$

The coefficients C_n of the series expansion for uniform wall heat flux are determined by the requirement that [3 4]

$$\sum_{n=1}^{\infty} C_nR_n(\eta) = -\left[\left(\eta^2 - \frac{1}{4}\eta^4 - \frac{7}{24}\right) - \left(\frac{1}{2}\eta^2 - \frac{1}{4}\eta^4\right)\left(\frac{u_s}{\bar{u}}\right) + \left(\frac{1}{24}\right)\left(\frac{u_s}{\bar{u}}\right)^2\right] \tag{13}$$

which, with the orthogonality property of the eigenfunctions, leads to

$$C_n = 1/[\lambda(\partial^2R/\partial\eta\partial\lambda)]_{\eta=1\,\lambda=\lambda} \tag{14}$$

Hence,

$$D_n \equiv C_nR\,(1) = -(16\alpha)/[E + (E^2/F) + F\gamma_n^2] \tag{15}$$

These expressions were derived on the assumption that λ_n is large and, consequently, are supposedly valid only in that limit However, the use of Eqs (12) and (15) gives values that appear to "fit" between results for continuum flow ($u_s = 0$) and for slug flow ($u_s/\bar{u} \to 1$) even for the values of n as small as 2, as can be seen from the comparison shown in Table 1 The results for continuum flow were obtained from expressions presented by Dzung,[2] whereas for slug flow the eigenvalues are obtained as the roots of $J_1[(2\lambda_n)^{1/2}] = 0$, where J_1 is a Bessel function of the first kind and of first order; the coefficients D_n are then obtained from the simple result $D_n = -1/\lambda_n$

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Table 1 Eigenvalues and coefficients

	$u_s/\bar{u} = 0$	$\frac{2}{5}$	$\frac{2}{3}$	1
$\lambda_1^{1/2}$	2 531		2 65	2 710
$\lambda_2^{1/2}$	4 578	4 71	4 75	4 955
$\lambda_3^{1/2}$	6 599	6 78	6 88	7 195
$\lambda_4^{1/2}$	8 610	8 81	9 00	9 425
D_1	−0 1985		−0 1670	−0 1360
D_2	−0 0693	−0 0594	−0 0515	−0 0406
D_3	−0 0365	−0 0306	−0 0247	−0 0194
D_4	−0 0230	−0 0217	−0 0145	−0 0113

It should be mentioned that the first few eigenvalues apparently become slightly inaccurate (i e , do not fit) as $(u_s/\bar{u} \to 0)$ Also, interactions between the velocity and temperature fields, such as thermal creep, have been neglected, as noted earlier

From this comparison it appears that the method of Sellars et al has definite application for determining the eigenvalues and coefficients for heat transfer in laminar tube slip flow The author is currently extending the treatment outlined here to the problem of heat transfer to laminar slip flow in a parallel plate channel

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Numerical Study of Hydrogen-Fluorine Kinetics in Nozzles

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A NUMBER of groups now have general computer programs for the study of complex chemical reactions in adiabatic flow [1–5] A program written here in FORTRAN language has been used to compare calculated and experimental performances in the H_2 F_2 system

The equations and techniques necessary have been amply described in the references cited Briefly, the usual hydrodynamic equations for one-dimensional, adiabatic, frictionless, ideal-gas flow were solved along with general equations in matrix form for coupled chemical reaction kinetics To begin the expansion from initial equilibrium, the chemical kinetic equations were linearized about the equilibrium composition and an analytical solution obtained by standard eigenvalue techniques The linear chemical kinetic equations were based on a straightforward generalization of the equations of Brokaw[6] for a single reaction When the departure from equilibrium became appreciable, the computation was switched to a conventional Runge-Kutta integration of the complete nonlinear kinetic equations Other approaches to the problem of starting the integration from an equilibrium state include starting with the nonlinear equations and a very small integration interval, after a small initial perturbation of the starting composition,[4] and another type of perturbation scheme which assumes equilibrium composition derivatives and density [1] It has been found here, as well as by other workers, that the initial portion of the computation is not critical as long as it can be carried out in some manner A possible advantage of linearization- or perturbation-type schemes would be the reduction in computing time for flows with substantial near-equilibrium portions Computing times on the IBM 7090 for the present work were in the range 5–20 min, depending on the chamber pressure and expansion ratio

A relatively simple approximate approach was used to find the mass flow rate and initial area The ratio of mass discharge rate to throat area has an eigenvalue nature[4 7] but does not vary markedly with initial area, velocity, or nature of the flow up to the throat Since the flow is usually near equilibrium from the chamber to the throat, it was decided to use throat area as the scale parameter and calculate the mass discharge rate w from a chosen throat area A_t and the equilibrium ratio of w/A_t, found from a conventional specific impulse program An arbitrary, small chamber velocity was assumed and the chamber area found from w Large variations in the initial velocity did not affect the subsequent quantities materially The actual throat area was always within 1% of the nominal area used initially If great accuracy were desired, an iterative procedure could have been used to find the true eigenvalue w/A_t

The following kinetic scheme involving HF, H_2, F, and H was used:

$$H + F + M = HF + M \quad (1)$$

$$2H + H_2 = 2H_2 \quad (2)$$

$$2H + H = H_2 + H \quad (3)$$

$$2H + HF = H_2 + HF \quad (4)$$

$$F + H_2 = HF + H \quad (5)$$

Since all mixtures studied were stoichiometric or hydrogen-rich, the concentration of F_2 was quite small and was neglected The recombination of H atoms was written with three different third bodies, since there are experimental data[8–10] on reactions (2) and (3) The forward rate constants were of the form

$$k_l = B_l T^{l} e^{-E_l/RT} \quad (6)$$

and the values used initially are given in Table 1

The values of B_2 and B_3 are the upper limits of the range of values obtained by different workers For example, values of B_2 of 3×10^{18} have been obtained by Rink[9] and $5\,6 \times 10^{18}$ by Avramenko and Kolesnikova [10] Preliminary work indicated that the calculated performance would be lower than the available measurements,[12] and so the highest values of the most recent data were used Over-all performance values were not very sensitive to the range of experimental values

A series of computed values for vacuum specific impulse I_{vac} is shown in Fig 1, along with the frozen, equilibrium, and

Table 1 Hydrogen-fluorine rate parameters

Reaction	B_l, cm³ mole-sec	v_l	E_l, kcal/mole	Source
1	$7\,5 \times 10^{18}$	−1	0	Assumed same as reaction (2)
2	$7\,5 \times 10^{18}$	−1	0	Ref 8
3	5×10^{19}	−1	0	Ref 8
4	$7\,5 \times 10^{18}$	−1	0	Assumed same as reaction (2)
5	5×10^{12}	0	5 7	By analogy with O + H_2 (Ref 11)

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